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TWO-PRIMARY ANALOGUES OF SELICK'S THEOREM AND THE KAHN–PRIDDY THEOREM FOR THE 3-SPHERE

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Dedicated to my friend and teacher John C. Moore

THE MAIN result of this paper is a 2-primary decomposition of a certain loop space analogous to that given in Theorem 4 of [17] for odd primes by Selick. A corollary is an analogue of the Kahn–Priddy theorem for the 3-sphere. Throughout this paper all spaces are localized at the prime 2 unless otherwise stated.

Let $2: \Omega X \to \Omega X$ denote the H-space squaring map for the based loop space ΩX. $(\Omega X)\{2\}$ denotes the homotopy theoretic fibre of 2. W_n is the homotopy theoretic fibre of the double suspension $E^2: S^{2n-1} \to \Omega^2 S^{2n+1}$, $S^3\langle 3\rangle$ is the 3-connected cover of S^3, and $\Omega_0^n S^n$ is the component of the base-point in $\Omega^n S^n$.

THEOREM 1.1. *$\Omega_0^3 S^3$ is a retract of $(\Omega^3 S^5)\{2\}$.*

The map giving the retraction is fairly explicit and has other applications such as Proposition 1.7. Sacrificing control over the retraction through modifications, one has

THEOREM 1.2. *$(\Omega^2 S^5)\{2\}$ is homotopy equivalent to $\Omega^2 S^3\langle 3\rangle \times W_2$.*

Since the fourth power map on $(\Omega^2 S^5)\{2\}$ is null-homotopic, these theorems give a (perhaps convoluted) proof of James' result that S^3 has exponent 4 [11; Theorem 1.20].

COROLLARY 1.3. *S^3 has exponent* 4.

Recall that $\pi_q(X; \mathbb{Z}/2)$ is the set of based homotopy classes of maps $\Sigma^{q-2}\mathbb{R}P^2$ to X.

COROLLARY 1.4. *If $q \geq 4$, then $\pi_q S^3$ is a direct summand of $\pi_{q+1}(S^5; \mathbb{Z}/2)$.*

One formulation of the Kahn–Priddy theorem [12] is that there is a map $\theta': \Omega^\infty\Sigma^\infty\mathbb{R}P^\infty \to \Omega_0^\infty S^\infty$ which induces a split epimorphism on homotopy groups.

COROLLARY 1.5. *There is a homotopy commutative diagram*

$$\begin{tikzcd}
(\Omega^3 S^5)\{2\} \arrow[r, "\theta"] \arrow[d, "\alpha"] & \Omega_0^3 S^3 \arrow[d, "E^\infty"] \\
\Omega^\infty\Sigma^\infty\mathbb{R}P^\infty \arrow[r, "\theta'"] & \Omega_0^\infty S^\infty
\end{tikzcd}$$

where E^∞ is the stabilization map and the two maps θ and θ' induce split epimorphisms on homotopy groups.

James constructs maps $H_2: \Omega S^{n+1} \to \Omega S^{2n+1}$ with (2-local) homotopy theoretic fibre S^n in Section 15 of [10]. With $n = 2$ Toda shows that the restriction of H_2 to $\Omega S^3\langle 3\rangle$ is a map $\hat{h}_2: \Omega S^3\langle 3\rangle \to \Omega S^5$ with homotopy theoretic fibre S^3 by Theorem 2.4 of [19]. Thus there is an induced fibration $i: S^3 \to \Omega S^3\langle 3\rangle$ with homotopy theoretic fibre $\Omega^2 S^5$ and an induced map $\Delta: \Omega^2 S^5 \to S^3$.

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COROLLARY 1.6. *There is a map* $\tau\colon \Omega S^3 \to \Omega^3 S^5$ *such that the diagram*

$$\begin{tikzcd} \Omega^3 S^5 \arrow[rr, "2"] \arrow[dr, "\Omega\Delta"'] & & \Omega^3 S^5 \\ & \Omega S^3 \arrow[ur, "\tau"'] & \end{tikzcd}$$

homotopy commutes.

We do not know whether the map τ induces the double suspension on homotopy groups.

Much of the work here involves various function spaces. One point is that several natural loop maps are composites of maps which are not even homotopy multiplicative. We include an example related to the Whitehead product and the "η-homomorphism" of G. W. Whitehead [22].

Let X^A denote the space of pointed maps from A to X for pointed spaces A and X. A pointed map $\alpha\colon B \to A$ induces $\alpha^*\colon X^A \to X^B$.

Let $\bar{\eta}\colon S^5 \to BS^3$ ($= \mathbb{H}P^\infty$) denote the generator of $\pi_5 BS^3 \cong \mathbb{Z}/2$. Let $\eta\colon S^3 \to S^2$ denote the Hopf map and $[\iota_n, \iota_n]$ in $\pi_{2n-1} S^n$ the Whitehead square. In Section 8 we define the map π: $\Omega S^9 \to BS^3$ used in the next proposition.

PROPOSITION 1.7. *There is a map* $\pi\colon \Omega S^9 \to BS^3$ *such that* $(\Omega^2 \pi)\cdot\eta^*$ *is homotopic to* $\Omega^2(\bar{\eta}\cdot[\iota_5, \iota_5])$. *Furthermore,* $\eta^*\colon \Omega^2 S^9 \to \Omega^3 S^9$ *is not homotopy multiplicative.*

Two homological computations are of use here.

PROPOSITION 1.8. *Let* $f\colon \Omega_0^3 S^3 \to \Omega_0^3 S^3$ *be any map which induces an isomorphism on* π_i, $i = 1, 2$. *Then* f *is a homotopy equivalence* (*at* 2).

The computations in Proposition 1.8 are used in [2] to prove that if f in Proposition 1.8 induces an isomorphism on π_1, then it induces an isomorphism on π_2. Thus $\Omega_0^3 S^3$ is "atomic." Furthermore it has been proven in Theorem 1.1 of [6] (respectively Theorem 5.1 of [1]) that if f is multiplicative (respectively homotopy multiplicative) and f induces an isomorphism on π_1, then f is an equivalence. One might wonder whether Proposition 1.8 remains true if 3 is replaced by $2n + 1$.

Our computations in Theorem 3.5 of [2] give

PROPOSITION 1.9. *Let* $f\colon X \to (\Omega^2 S^{2n+1})\{2\}$ *be a map which induces a mod-2 homology isomorphism on the module of primitives in dimensions* $2n - 2$ *and* $4n - 3$ *for* $n \geq 2$. *If the mod-2 homology of* X *is isomorphic to that of* $(\Omega^2 S^{2n+1})\{2\}$ *as a coalgebra over the Steenrod algebra, then* f_* *is an isomorphism.*

In this paragraph we digress for a moment to odd primes p. Selick proves that $\Omega^2 S^3 \langle 3 \rangle$ is a retract of $(\Omega^2 S^{2p+1})\{p\}$; Theorems 1.1 and 1.2 are analogues of the main results of [16, 17]. The methods here are somewhat different because S^{2p+1} fails to be a p-local H-space precisely when $p = 2$. In addition, the proofs of Corollaries 1.5 and 1.6 apply if one replaces "2" by "p". In particular it follows that the p^{th} power map on $\Omega^3 S^{2p+1}$ factors through Toda's map $\Omega\Delta\colon \Omega^3 S^{2p+1} \to \Omega S^{2p-1}$. Remarks on proofs are given in Section 2.

This paper is organized as follows. Section 1 summarizes the main results. Proofs of the main results are given in Sections 2 through 4. Applications of H-space deviations and co-H-space codeviations are given in Section 5. Sections 6 through 12 give proofs of the requisite technical results while Sections 13 through 15 give the relevant homological computations.

Most of the work here is in proving Theorem 1.1. Much of the analogous work required to prove Theorem 1.2 is given in [2] and [8] from which we quote liberally.

§2. PROOFS OF 1.3 TO 1.6

$(\Omega X)\{2\}$ is homotopy equivalent to X^A the function space of pointed maps from A to X where A is $\mathbb{R}P^2$. Thus $(\Omega^2 S^5)\{2\}$ is homotopy equivalent to $(S^5)^{\Sigma \mathbb{R}P^2}$. The suspension order of the identity of $\Sigma\mathbb{R}P^2$ is 4 by Theorem 4.1 of [20]. Thus the fourth power map on $(\Omega^2 S^5)\{2\}$ is null and so Corollary 1.3 follows.

Similarly $\pi_q(\Omega^3 S^5)\{2\}$ is isomorphic to $\pi_{q+4}(S^5; \mathbb{Z}/2)$. Thus Corollary 1.4 follows.

Let $E^\infty\colon \Omega^n\Sigma^n X \to \Omega^\infty\Sigma^\infty X$ denote the stabilization map. Let $i\colon \mathbb{R}P^n \to \mathbb{R}P^\infty$ denote the standard inclusion. By the main result of [12] and Corollary 3.2 of [7] there is a homotopy commutative diagram

$$\begin{tikzcd}
\Omega_0^n S^n \arrow[r, "h_2"] \arrow[d, "E^\infty"'] & \Omega^\infty\Sigma^\infty \mathbb{R}P^{n-1} \arrow[d, "\Omega^\infty\Sigma^\infty(i)"] \\
\Omega_0^\infty S^\infty \arrow[r, "h_2"] & \Omega^\infty\Sigma^\infty\mathbb{R}P^\infty
\end{tikzcd}$$

and a map $\theta'\colon \Omega^\infty\Sigma^\infty\mathbb{R}P^\infty \to \Omega_0^\infty S^\infty$ such that $\theta'\cdot h_2$ is homotopic to the identity.

Let $\theta\colon (\Omega^3 S^5)\{2\} \to \Omega_0^3 S^3$ be any choice of retraction in Theorem 1.1. Define $\alpha\colon (\Omega^3 S^5)\{2\} \to \Omega^\infty\Sigma^\infty\mathbb{R}P^\infty$ to be the composite $\Omega^\infty\Sigma^\infty(i)\cdot h_2\cdot\theta$. Thus $\theta'\cdot\alpha$ is homotopic to $E^\infty\cdot\theta$ and Corollary 1.5 follows.

Again by [12] and [17], it follows that there is a p-local homotopy commutative diagram, $p \geq 2$,

$$\begin{tikzcd}
\Omega_0^n S^n \arrow[r, "h_p"] \arrow[d, "E^\infty"'] & \Omega^\infty\Sigma^\infty B(R^n, p) \arrow[d, "\Omega^\infty\Sigma^\infty(i)"] \\
\Omega_0^\infty S^\infty \arrow[r, "h_p"] & \Omega^\infty\Sigma^\infty B(R^\infty, p)
\end{tikzcd}$$

and a map $\theta'\colon \Omega^\infty\Sigma^\infty B(R^\infty, p) \to \Omega_0^\infty S^\infty$ such that $\theta'\cdot h_p$ is homotopic to the identity where $B(R^n, p)$ is the configuration space of p-tuples of distinct points in $\mathbb{R}^n$. Define f: $(\Omega^3 S^{2p+1})\{p\} \to \Omega^\infty\Sigma^\infty B(R^\infty, p)$ as in the above paragraph and observe that the odd primary analogue of Corollary 1.5 follows.

Before proving Corollary 1.6, we remark that by Lemma 3.1 of [3], it follows that there exists a homotopy commutative diagram

$$\begin{tikzcd}
\Omega^2 S^{2n+1} \arrow[rr, "2"] \arrow[dr, "\delta"'] & & \Omega^2 S^{2n+1} \\
& \Omega S^{2n} \arrow[ur, "\Omega E"'] &
\end{tikzcd}$$

The content of Corollary 1.6 is that setting $n = 2$ one obtains a better result after looping.

In the proofs of Theorems 1.1 and 1.2 we construct maps $h\colon \Omega^2 S^3 \to (\Omega^2 S^5)\{2\}$ such that the composite $j\cdot h\colon \Omega^2 S^3 \to \Omega^2 S^5$ is homotopic to ΩH_2 where H_2 is the second Hilton-Hopf invariant and $j\colon (\Omega^2 S^5)\{2\} \to \Omega^2 S^5$ is the natural map. The retractions in Theorems 1.1 and 1.2 are obtained from retractions for the map h when restricted to $\Omega^2 S^3\langle 3\rangle$. Thus by Theorem 1.2 there is a homotopy commutative diagram

$$\begin{tikzcd}
\Omega^3 S^5 \arrow[r, "i"] \arrow[d, "1"'] & (\Omega^2 S^5)\{2\} \arrow[d, "\gamma"] \\
\Omega^3 S^5 \arrow[r, "\gamma i"] & Z
\end{tikzcd}$$

where the homotopy theoretic fibre of γ is $\Omega^2 S^3\langle 3\rangle$ and i is the natural map. Of course, Z

has the homotopy type of W_2 by Theorem 1.2, but we do not need this. In any case, there is a morphism of fibrations

$$\begin{tikzcd}
\Omega_0^3 S^3 \arrow[r] \arrow[d, "\Omega^2 H_2"'] & * \arrow[r] \arrow[d] & \Omega^2 S^3 \langle 3 \rangle \arrow[d] \\
\Omega^3 S^5 \arrow[r, "2"] \arrow[d] & \Omega^3 S^5 \arrow[r, "i"] \arrow[d, "1"] & (\Omega^2 S^5)\{2\} \arrow[d, "\gamma"] \\
X \arrow[r] & \Omega^3 S^5 \arrow[r, "\gamma i"] & Z
\end{tikzcd} .$$

Thus X is homotopy equivalent to the fibre of the map $\Omega H_2 \colon \Omega^2 S^3 \langle 3 \rangle \to \Omega^2 S^5$ and is thus homotopy equivalent to ΩS^3. Hence the map 2: $\Omega^3 S^5 \to \Omega^3 S^5$ factors through ΩS^3.

To obtain the analogous result for $p > 2$, replace $(\Omega^2 S^5)\{2\}$ by $(\Omega^2 S^{2p+1})\{p\}$ and use P. Selick's Theorem 4 in [17].

§3. PROOF OF 1.1.

The proofs of Theorems 1.1 and 1.2 follow the same general pattern and use similar maps. Let $[X, Y]$ denote the set of pointed homotopy classes of maps from X to Y. $[\Sigma X, Y]$ and $[X, \Omega Y]$ are naturally isomorphic groups. The k^{th} power map on ΣX induced by the suspension structure will be written $[k]$.

Consider the Hilton-Hopf invariant $H_2 \colon \Omega S^{2n+1} \to \Omega S^{4n+1}$. The following lemma is due to M. G. Barratt; a proof is given in Lemma 3.1 of [3].

LEMMA 3.1. *ΩH_2 has order 2 in the abelian group $[\Omega^2 S^{2n+1}, \Omega^2 S^{4n+1}]$ and thus there is a lift h of ΩH_2 to $(\Omega^2 S^{4n+1})\{2\}$. This lift induces an isomorphism on $H_{4n-2}(\ ;\mathbb{Z}/2)$.*

The proof of Lemma 3.1 in [3] applies to maps $h_2 \colon \Omega S^{2n+1} \to \Omega S^{4n+1}$ where the diagrams

(i)
$$\begin{tikzcd}
\Omega S^{2n+1} \arrow[r, "h_2"] \arrow[d, "\Omega[-1]"'] & \Omega S^{4n+1} \arrow[d, "1"] \\
\Omega S^{2n+1} \arrow[r, "h_2"] & \Omega S^{4n+1}
\end{tikzcd}$$
, and

(ii)
$$\begin{tikzcd}
\Omega S^{4n+1} \arrow[r, "h_2"] \arrow[d, "\Omega\Sigma\beta"'] & \Omega S^{8n+1} \arrow[d, "\Omega\Sigma(\beta \wedge \beta)"] \\
\Omega S^{2n+1} \arrow[r, "h_2"] & \Omega S^{4n+1}
\end{tikzcd}$$

homotopy commute for $\beta \colon S^{4n} \to S^{2n}$. Thus h_2 may be any sort of reasonable Hopf invariant and Lemma 3.1 remains valid.

Define $h \colon \Omega^2 S^3 \langle 3 \rangle \to (\Omega^2 S^5)\{2\}$ to be the lift in Lemma 3.1 restricted to $\Omega^2 S^3 \langle 3 \rangle$ where $n = 1$. There are many choices of h, none of which are H-maps.

To prove Theorem 1.1, it suffices by Proposition 1.8 to construct a map θ: $(\Omega^3 S^5)\{2\} \to \Omega_0^3 S^3$ such that $\theta \cdot \Omega h$ induces an isomorphism on π_i, $i \leq 2$. The crux of the construction of θ is

THEOREM 3.2. *There exists a map $g \colon \Omega^2 S^5 \to \Omega S^3$ such that*

(i) g induces an epimorphism on π_i, $i = 3, 4$, and

(ii) g has order 2 in the abelian group $[\Omega^2 S^5, \Omega S^3]$.

Example 3.3. Let $\bar{\eta}\colon S^5 \to BS^3$ denote the generator of $\pi_5 BS^3 \cong \mathbb{Z}/2$. Then $\Omega^2\bar{\eta}$ has order 4 in the abelian group $[\Omega^2 S^5, \Omega S^3]$. This was essentially proven in Example 1.3 of [4] and a proof is given in Section 8.

In any case, granting Theorem 2.2 we give the proof of Theorem 1.1. Since Ωg is multiplicative, the diagram

$$\begin{tikzcd}
\Omega^3 S^5 \arrow[r] \arrow[d, "2"'] & * \arrow[d] \\
\Omega^3 S^5 \arrow[r, "\Omega g"] & \Omega^2 S^3
\end{tikzcd}$$

homotopy commutes by Theorem 3.2(ii). Thus there is a lift

$$\theta\colon\ (\Omega^3 S^5)\{2\} \to \Omega_0^3 S^3$$

and a morphism of fibrations

$$\begin{tikzcd}
(\Omega^3 S^5)\{2\} \arrow[r, "\theta"] \arrow[d] & \Omega_0^3 S^3 \arrow[d] \\
\Omega^3 S^5 \arrow[r] \arrow[d, "2"'] & * \arrow[d] \\
\Omega^3 S^5 \arrow[r, "\Omega g"'] & \Omega^2 S^3 .
\end{tikzcd}$$

Consider the morphism of long exact homotopy sequences

$$\begin{tikzcd}[column sep=small]
\pi_1(\Omega^3 S^5)\{2\} \arrow[d] & \pi_2\Omega^3 S^5 \arrow[l, "\partial"'] \arrow[d] & \pi_2\Omega^3 S^5 \arrow[l, "2_*"'] \arrow[d] & \pi_2(\Omega^3 S^5)\{2\} \arrow[l] \arrow[d] & \pi_3\Omega^3 S^5 \arrow[l, "\partial"'] \arrow[d] & \pi_3\Omega^3 S^5 \arrow[l, "2_*"'] \arrow[d] \\
\pi_1\Omega_0^3 S^3 & \pi_2\Omega^2 S^3 \arrow[l, "\cong", "\partial"'] & 0 \arrow[l] & \pi_2\Omega_0^3 S^3 \arrow[l] & \pi_3\Omega^2 S^3 \arrow[l, "\cong", "\partial"'] & 0. \arrow[l]
\end{tikzcd}$$

Since $\pi_2\Omega^3 S^5 \cong \mathbb{Z}$, $\pi_3\Omega^3 S^5 \cong \mathbb{Z}/2$, and 2_* induces multiplication by 2, it follows from Theorem 3.2(ii) that θ induces an isomorphism on π_i, $i = 1, 2$.

A similar check on homotopy groups for the morphism of fibrations

$$\begin{tikzcd}
\Omega^2 S^3 \arrow[r, "h"] \arrow[d, "1"] & (\Omega^2 S^5)\{2\} \arrow[d] \\
\Omega^2 S^3 \arrow[r, "\Omega h_2"] \arrow[d] & \Omega^2 S^5 \arrow[d, "2"] \\
* \arrow[r] & \Omega^2 S^5
\end{tikzcd}$$

gives that h induces an isomorphism on π_i, $i = 2, 3$. That is, there is a lift σ in the diagram

$$\begin{tikzcd}
S^2 \arrow[r, "\sigma"] \arrow[d, "\hat{\eta}_3"'] & \Omega^3 S^5 \arrow[d] \\
\Omega^2 S^3 \arrow[r, "h"] \arrow[dr, "\Omega h_2"'] & (\Omega^2 S^5)\{2\} \arrow[d] \\
& \Omega^2 S^5
\end{tikzcd}$$

where σ is of odd degree, and $\hat{\eta}_3$ generates $\pi_2\Omega^2S^3$. Hence σ_*: $\pi_3S^2 \to \pi_3\Omega^3S^5$ is an epimorphism.

Thus Theorem 1.1 follows from Proposition 1.8

We shall show in Proposition 6.2 that g cannot be chosen to be an H-map.

§4. PROOF OF 1.2.

The proof of Theorem 1.2 uses the map h: $\Omega^2S^3\langle 3\rangle \to (\Omega^2S^5)\{2\}$ given in Section 3. Using the methods in the proof of Theorem 3.2 we show in [8]

THEOREM 4.1. *There is a map* $\bar{\rho}$: $\Omega^2S^9 \to \Omega S^5$ *which induces an epimorphism on* π_7.

Since $\Omega(\bar{\rho})$ is multiplicative, there is an induced map ρ: $(\Omega^3S^9)\{2\} \to (\Omega^2S^5)\{2\}$. In Proposition 1.3 of [3] we constructed maps σ_n: $W_n \to (\Omega^3S^{4n+1})\{2\}$ which induce isomorphisms on H_{4n-2}. Let σ denote the composite $\rho\cdot\sigma_2$. Define γ: $\Omega^2S^3\langle 3\rangle \times W_2 \to (\Omega^2S^5)\{2\}$ to be the composite

$$\Omega^2S^3\langle 3\rangle \times W_2 \xrightarrow{h\times\sigma} (\Omega^2S^5)\{2\} \times (\Omega^2S^5)\{2\} \xrightarrow{\text{multiply}} (\Omega^2S^5)\{2\}.$$

We require

LEMMA 4.2. (*i*) γ *induces an isomorphism on the module of primitives for mod-*2 *homology in degrees* 2 *and* 5.

(*ii*) *The mod-*2 *homology of* $\Omega^2S^3\langle 3\rangle \times W_2$ *is isomorphic to that of* $(\Omega^2S^5)\{2\}$ *as a coalgebra over the Steenrod algebra.*

Thus Theorem 1.2 follows from Lemma 4.2 and Proposition 1.9.

§5. DEVIATIONS, CODEVIATIONS, AND LOOP MAPS

To prove the main technical results (Theorems 3.2 and 4.1), one needs to know that certain maps are homotopic to loop maps. Throughout this section assume that X stands for a connected CW-complex. The observations in this section are well-known. Proofs are included for convenience.

LEMMA 5.1. *An H-map* f: $\Omega\Sigma X \to \Omega Y$ *is homotopic to a loop map.*

Proof. Let g: $\Omega\Sigma X \to \Omega Y$ denote the canonical loop map obtained by the unique multiplicative extension of the composite $X \xrightarrow{E} \Omega\Sigma X \xrightarrow{f} \Omega Y$ where E is the suspension. g is $\Omega(\bar{g})$ where $\bar{g}$ is the adjoint of $f\cdot E$. We claim that g is homotopic to f.

Let μ_n: $X^n \to \Omega\Sigma X$ be given by the multiplication in a fixed order [for example, $\mu_n(x_1, \ldots, x_n) = x_1(x_2(\ldots(x_{n-1}x_n)\ldots))$]. By hypothesis, f is homotopy multiplicative and so $f\cdot\mu_n$ is homotopic to $g\cdot\mu_n$.

Let $\prod_n X^n$ denote the weak direct product: that is, the direct limit of finite products. Let μ: $\prod_n X^n \to \Omega\Sigma X$ be given by the maps μ_n. The induced group homomorphism

$$\mu^*: [\Omega\Sigma X, \Omega Y] \to [\textstyle\prod_n X^n, \Omega Y]$$

is a split monomorphism of sets since $\Sigma\Omega\Sigma X$ is a wedge summand of $\Sigma(\prod_n X^n)$. Since $f\cdot\mu$ and $g\cdot\mu$ are homotopic, so are f and g, and the lemma follows.

Let α: $\Sigma A \to \Sigma B$ be any pointed map and define the codeviation

$$D(\alpha): \Sigma A \to \Sigma B \vee \Sigma B$$

to be $(\alpha\vee\alpha)\psi_A - \psi_B\cdot\alpha$ where ψ_X denotes the natural comultiplication in ΣX and "$-$" is subtraction in the group $[\Sigma A, \Sigma B\vee\Sigma B]$. Let i: $\Sigma B\vee\Sigma B \to \Sigma B\times\Sigma B$ denote the natural inclusion. Evidently $iD(\alpha)$ is null. Hence $D(\alpha)$ lifts to the homotopy theoretic fibre of i. Of course one could replace ΣA by a cogroup or ΣB by a co-H space in the above, but we elect to do otherwise. In any case, the homotopy theoretic fibre of i is homotopy equivalent to $\Sigma_{i,j\geq 1} \vee B^{[i+j]}$ if B is connected where $B^{[n]}$ is the n-fold smash product of B by the corollary to Theorem 3 of [13]. This codeviation was studied in [4]. Next, notice that the codeviation $D(\alpha)$ factors as

$$\begin{tikzcd} & \Sigma\,(\vee_{i,j\geq 1} B^{[i+j]}) \arrow[d, "\varepsilon"] \\ \Sigma A \arrow[ur, "\tilde{D}(\alpha)"] \arrow[r, "D(\alpha)"'] & \Sigma B \vee \Sigma B \end{tikzcd}$$

where ε is a sum of maps w_{i+j}: $\Sigma B^{[i+j]} \to \Sigma B \vee \Sigma B$ which are generalized Whitehead products.

Evidently, α induces a map $\alpha^*: X^{\Sigma B} \to X^{\Sigma A}$ and $X^{\Sigma A}$ is homotopy equivalent to $\Omega(X^A)$. α^* may not necessarily be homotopy multiplicative. Notice that $D(\alpha)$ induces

$$D(\alpha)^*: X^{\Sigma B} \times X^{\Sigma B} \to X^{\Sigma A}.$$

Clearly, we have

LEMMA 5.2. *α^* is an H-map if and only if $D(\alpha)^*$ is null-homotopic.*

This information will be applied to $\eta: S^3 \to S^2$ which of course is not a co-H map. Indeed, the "hom dual" $\eta^*: \Omega^2 X \to \Omega^3 X$ is in general not an H-map. (However if X is an H-space, then η^* is an H-map.) In particular, we need the specific form of $D(\eta)^*$ to use in the proof of Theorem 7.3. First, $D(\eta)$ must be computed. Let W denote the Whitehead product $[\iota_1, \iota_2]$ where ι_j: $S^2 \to S^2 \vee S^2$, $j = 1, 2$, are the natural inclusions.

LEMMA 5.3. *$D(\eta)$: $S^3 \to S^2 \vee S^2$ is the Whitehead product $\pm[\iota_1, \iota_2]$.*

Proof. Consider the inclusion $S^2 \vee S^2 \to S^2 \times S^2$ with homotopy fibre $\Sigma(\Omega S^2 \wedge \Omega S^2)$. Furthermore, $S^3 \xrightarrow{[\iota_1,\iota_2]} S^2 \vee S^2 \to S^2 \times S^2$ is a cofibration. Thus $[\iota_1, \iota_2]$ lifts to $\Sigma(\Omega S^2 \wedge \Omega S^2)$ and this lift induces an integral homology isomorphism in degrees less than 4 by the Serre exact sequence. Thus, the codeviation $D(\eta)$ factors as

$$\begin{tikzcd} & S^3 \arrow[d, "{[\iota_1,\iota_2]}"] \\ S^3 \arrow[ur, "\rho"] \arrow[r, "D(\eta)"'] & S^2 \vee S^2 \end{tikzcd}$$

up to homotopy and so it suffices to compute the degree of ρ.

LEMMA 5.4. *ρ is of degree ± 1.*

Proof. Adjoint $D(\eta)$ to give $\overline{D(\eta)}: S^2 \to \Omega(S^2 \vee S^2)$. By definition, $\overline{D(\eta)}$ is the loop sum (in the group $[S^2, \Omega(S^2 \vee S^2)]$) of the following two maps

(i) $S^2 \xrightarrow{\overline{\nabla}} \Omega(S^3 \vee S^3) \xrightarrow{\Omega(\eta \vee \eta)} \Omega(S^2 \vee S^2)$, and

(ii) $S^2 \xrightarrow{E} \Omega S^3 \xrightarrow{\Omega\eta} \Omega S^2 \xrightarrow{\Omega(\nabla)} \Omega(S^2 \vee S^2) \xrightarrow{-1} \Omega(S^2 \vee S^2)$

where $\overline{\nabla}$ is the adjoint of the comultiplication ∇: $S^n \to S^n \vee S^n$ and -1 is the loop inverse.

We compute the maps in (i) and (ii) in integral homology. First, recall that $H_*\Omega(S^3 \vee S^3) \cong T[y_1, y_2]$, $H_*\Omega S^2 \cong T[x]$ and $H_*\Omega(S^2 \vee S^2) \cong T[x_1, x_2]$ with x_i and y_i induced by the natural inclusions, and $T[V]$ denotes the tensor algebra of the free abelian group V. Let ι denote the fundamental class of S^2. Then

(iii) $\Omega(\eta \vee \eta)_*(y_i) = x_i^2$, and so

(iv) $\Omega(\eta \vee \eta)_* \overline{\nabla}_*(\iota) = x_1^2 + x_2^2$.

Similarly

(v) $(\Omega\eta)_* E_*(\iota) = x^2$, and so

(vi) $\Omega(\nabla)_*(\Omega\eta)_* E_*(\iota) = (x_1 + x_2)^2$.

Thus

$$\overline{D(\eta)_*}(\iota) = x_1^2 + x_2^2 - (x_1 + x_2)^2$$

and so $\overline{D(\eta)_*}(\iota) = -[x_1, x_2]$ in $H_*\Omega(S^2 \vee S^2)$. Hence ρ is of degree ± 1 and the lemma follows.

Consider

$$\Omega^2 X \times \Omega^2 X \xrightarrow{W^*} \Omega^3 X$$

where W^* is the "hom dual" of $W = [\iota_1, \iota_2]$. Observe that W^* factors through $(\Omega^2 X) \wedge (\Omega^2 X)$. By abuse of notation, let W^* denote the induced map $W^*: (\Omega^2 X) \wedge (\Omega^2 X) \to \Omega^3 X$.

COROLLARY 5.5. *The H-deviation of* η^*: $\Omega^2 X \to \Omega^3 X$, $D(\eta)^*$, *factors through* W^*: $\Omega^2 X \wedge \Omega^2 X \to \Omega^3 X$.

Next consider the composite

$$Y \wedge Y \xrightarrow{E^2 \wedge E^2} (\Omega^2 \Sigma^2 Y) \wedge (\Omega^2 \Sigma^2 Y) \xrightarrow{W^*} \Omega^3 (\Sigma^2 Y)$$

together with the Samelson product

$$[E, E]: (\Sigma Y) \wedge (\Sigma Y) \to \Omega\Sigma(\Sigma Y)$$

where E: $\Sigma Y \to \Omega\Sigma(\Sigma Y)$ denotes the suspension. Finally consider the twist map τ: $S^1 \wedge Y \to Y \wedge S^1$. The following proposition will be used later.

PROPOSITION 5.6. $W^* \cdot (E^2 \wedge E^2)$ *and* $[E, E] \cdot (1 \wedge \tau \wedge 1)$ *are adjoints.*

Proof. The appropriate adjoint of $W^* \cdot (E^2 \wedge E^2)$, f, is given by the composite

$$\begin{tikzcd}
S^2 \wedge Y \wedge Y \arrow[r, "1 \wedge E^2 \wedge E^2"] \arrow[drr, "f"'] & S^2 \wedge (\Omega^2 \Sigma^2 Y) \wedge (\Omega^2 \Sigma^2 Y) \arrow[r, "1 \wedge W^*"] & S^2 \wedge \Omega^3 (\Sigma^2 Y) \arrow[d, "\text{evaluation}"] \\
 & & \Omega(\Sigma^2 Y)
\end{tikzcd}$$

A check on the point set level gives that $f \cdot (1 \wedge \tau \wedge 1)$ is homotopic to the Samelson product: In particular W may be chosen adjoint to the Samelson product so that

$$f \cdot (1 \wedge \tau \wedge 1)(s, t, x, y) = (s, x)(t, y)(-s, x)(-t, y)$$

for $(s, t, x, y) \in S^1 \wedge S^1 \wedge Y \wedge Y$. That $f \cdot (1 \wedge \tau \wedge 1)$ is homotopic to $[E, E]$ follows immediately from the observation that the diagram

$$\begin{tikzcd}
\Sigma Y \arrow[r, "E"] \arrow[d, "{[-1]}"'] & \Omega\Sigma^2 Y \arrow[d, "-1"] \\
\Sigma Y \arrow[r, "E"] & \Omega\Sigma^2 Y
\end{tikzcd}$$

homotopy commutes.

§6. CONSTRUCTION OF g AND THE PROOF OF 3.2

Consider the map $\Omega^2 \bar{\eta}$ in Example 3.3. Although $2\Omega^2\bar{\eta}$ is non-zero, we shall show

PROPOSITION 6.1. *There exists a map $\phi:\Omega^2 S^5 \to \Omega S^3$ such that ϕ induces the trivial map on π_i, $i \le 5$, and such that $2\phi = 2\Omega^2\bar{\eta}$ in $[\Omega^2 S^5, \Omega S^3]$.*

Let g denote the map in Theorem 3.2.

PROPOSITION 6.2. *Neither ϕ nor g can be chosen to be an H-map.*

Construction 6.3. $g = \Omega^2\bar{\eta} - \phi$ in $[\Omega^2 S^5, \Omega S^3]$ where ϕ is the map given in Proposition 6.1.

Since ϕ is trivial on π_i, $i \le 5$, and $\Omega^2\bar{\eta}$ induces an epimorphism on π_i, $i = 3, 4$, it follows that g induces an epimorphism on π_i, $i = 3, 4$.

Since $2\phi = 2\Omega^2\bar{\eta}$ in $[\Omega^2 S^5, \Omega S^3]$ by Proposition 6.1, the equation

$$2g = 2(\Omega^2\bar{\eta} - \phi) = 2\Omega^2\bar{\eta} - 2\phi = 0$$

is satisfied in $[\Omega^2 S^5, \Omega S^3]$. Thus Theorem 3.2. follows

To construct ϕ we must consider several maps. First consider the 2-local fibration $S^4 \xrightarrow{E} \Omega S^5 \xrightarrow{h_2} \Omega S^9$ which induces the EHP sequence [11] and thus obtain a fibration

$$\begin{tikzcd} \Omega^2 S^9 \arrow[r, "\partial"] & S^4 \arrow[d, "E"] \\ & \Omega S^5. \end{tikzcd}$$

Next, consider the map i: $S^4 \to BS^3$ where i is the inclusion of the bottom cell in the classifying space BS^3 $(= HP^\infty)$. Recall that $[k]$ denotes the degree k map on a suspension ΣX. Finally, consider the composite given by commutativity of the diagram

$$\begin{tikzcd} \Omega S^9 \arrow[r, "\Omega E"] \arrow[dr] & \Omega^2 S^{10} \arrow[d, "\Omega^2\eta_9"] \\ & \Omega^2 S^9 \end{tikzcd}$$

together with

LEMMA 6.4. *The diagram*

$$\begin{tikzcd} \Omega S^n \arrow[r, "\Omega E"] \arrow[d, "2"'] & \Omega^2 S^{n+1} \arrow[d, "\Omega^2[2]"] \\ \Omega S^n \arrow[r] & \Omega^2 S^{n+1} \end{tikzcd}$$

homotopy commutes.

Thus by Lemma 6.4, the composite

$$\Omega S^9 \xrightarrow{2} \Omega S^9 \xrightarrow{\Omega E} \Omega^2 S^{10} \xrightarrow{\Omega^2\eta_9} \Omega^2 S^9$$

is null-homotopic because $\Omega^2\eta_9 \cdot \Omega^2[2]$ is null. We obtain a morphism of fibration sequences

$$\begin{tikzcd} (\Omega S^9)\{2\} \arrow[r, "\alpha"] \arrow[d] & \Omega^3 S^9 \arrow[d] \\ \Omega S^9 \arrow[r] \arrow[d, "2"'] & * \arrow[d] \\ \Omega S^9 \arrow[r, "\Omega^2\eta_9 \cdot \Omega E"] & \Omega^2 S^9 \end{tikzcd}$$

where α is clearly not uniquely defined.

Using the maps above, we can now define ϕ.

Construction 6.5. ϕ is the composite

$$\Omega^2 S^5 \xrightarrow{h} (\Omega^2 S^9)\{2\} \xrightarrow{\Omega\alpha} \Omega^4 S^9 \xrightarrow{\Omega^2\partial} \Omega^2 S^4 \xrightarrow{\Omega^2 i} \Omega S^3$$

where α, ∂, and i are given above and h is the map in Lemma 3.1.

We start to prove Proposition 6.1. The proofs of Proposition 6.2 and Lemma 6.4 are easy and relegated to Section 11.

§7. 2ϕ

A "computation" of 2ϕ is necessary to do Proposition 6.1. In the construction (6.5) of ϕ, all maps except h are loop maps. Thus 2ϕ in $[\Omega^2 S^5, \Omega S^3]$ is given by the composite

$$\Omega^2 S^5 \xrightarrow{h} (\Omega^2 S^9)\{2\} \xrightarrow{2} (\Omega^2 S^9)\{2\} \xrightarrow{\xi} \Omega S^3$$

where ξ is the composite $(\Omega^2 i)\cdot(\Omega^2\partial)\cdot(\Omega\alpha)$.

In order to expand the map 2, we consider contravariant η: The Hopf map η: $S^3 \to S^2$ induces η^*: $\Omega^2 X \to \Omega^3 X$. Next consider the cofibration sequence $S^2 \xrightarrow{i} P^3(2) \xrightarrow{j} S^3$ where $P^n(k)$ is the cofibre of the degree k map on S^{n-1} and j is the pinch map. Thus there is an induced fibration sequence

$$\Omega^3 X \xrightarrow{j^*} X^{P^3(2)} \xrightarrow{i^*} \Omega^2 X.$$

Lemma 7.1. *There is a homotopy commutative diagram*

$$\begin{tikzcd}
(\Omega^2 X)\{2\} \arrow[r, "i^*"] \arrow[drr, "2"'] & \Omega^2 X \arrow[r, "\eta^*"] & \Omega^3 X \arrow[d, "j^*"] \\
& & (\Omega^2 X)\{2\}.
\end{tikzcd}$$

Proof. $(\Omega^2 X)\{2\}$ is homotopy equivalent to the function complex $(X)^{P^3(2)}$. The *H-space* squaring map on $(X)^{P^3(2)}$ is induced by the degree 2 map, [2], on $P^3(2)$. By Lemma 2.1 of [20] there is a homotopy commutative diagram

$$\begin{tikzcd}
P^3(2) \arrow[r, "j"] \arrow[drr, "{[2]}"'] & S^3 \arrow[r, "\eta"] & S^2 \arrow[d, "i"] \\
& & P^3(2)
\end{tikzcd}$$

and so Lemma 7.1 follows by taking "hom duals."

Next, we give

Construction 7.2. λ: $\Omega S^9 \to S^3$ is the composite

$$\Omega S^9 \xrightarrow{\Omega E} \Omega^2 S^{10} \xrightarrow{\Omega^2\eta_9} \Omega^2 S^9 \xrightarrow{\eta^*} \Omega^3 S^9 \xrightarrow{\Omega(i\partial)} S^3.$$

Although η^* is not homotopy multiplicative, we shall prove the following theorem in Section 9.

Theorem 7.3. *λ is an H-map and is thus homotopic to a loop map by Lemma* 5.1.

The composite $\Omega(\partial)\cdot\eta^*\cdot\Omega^2\eta_9\cdot\Omega E$ is not an H-map as remarked at the end of Section 9. We use Theorem 7.3 to prove

Corollary 7.4. *λ is homotopic to the loop map $\Omega(\overline{v'\eta_6\eta_7})$ where $\overline{v'\eta_6\eta_7}$: $S^9 \to BS^3$* represents the generator of $\pi_9 BS^3 \cong \mathbb{Z}/2$.

Proof: First compute λ on π_8. Notice that $\pi_8 S^3 \cong \mathbb{Z}/2$ with generator $v'\eta_6\eta_7$ [19; p. 44] and that λ on π_8 is given by the composite

$$S^8 \xrightarrow{\eta_7} S^7 \xrightarrow{\eta_6} S^6 \xrightarrow{E^3} \Omega^3 S^9 \xrightarrow{\Omega\partial} \Omega S^4 \xrightarrow{\Omega i} S^3.$$

But $(\Omega\partial)(\iota_6) = 2v_4 - E(v')$ where ι_6 is the fundamental class of $\Omega^3 S^9$ [19; p. 43]. Then $(2v_4 - E(v'))\eta_6\eta_7 = -E(v')\eta_6\eta_7$ which projects to $v'\eta_6\eta_7$ in $\pi_8 S^3$. Since $v'\eta_6\eta_7$ is the generator of $\pi_8 S^3$, λ and $\Omega(\overline{v'\eta_6\eta_7})$ agree on the bottom cell. Corollary 7.4 now follows from Lemma 5.1 and Theorem 7.3.

PROPOSITION 7.5. *There is a homotopy commutative diagram*

$$\begin{tikzcd}
\Omega^2 S^5 \arrow[d, "h"] \arrow[ddr, "2\phi"] & \\
(\Omega^2 S^9)\{2\} \arrow[d, "i^*"] & \\
\Omega^2 S^9 \arrow[r, "\Omega\lambda"] & \Omega S^3
\end{tikzcd}$$

Proof: By Lemma 7.1, there is a homotopy commutative diagram

$$\begin{tikzcd}
\Omega^2 S^5 \arrow[r, "h"] \arrow[ddd, "2\phi"] & \Omega^2 S^9\{2\} \arrow[r, "i^*"] & \Omega^2 S^9 \arrow[d, "\eta^*"] \\
 & & \Omega^3 S^9 \arrow[dd, "j^*"] \\
 & & \\
\Omega S^3 & \Omega^4 S^9 \arrow[l, "\Omega^2(i\partial)"] & \Omega^2 S^9\{2\} \arrow[l, "\Omega(\alpha)"]
\end{tikzcd}$$

By construction, the diagram

$$\begin{tikzcd}
\Omega^3 S^9 \arrow[r, "\Omega^3 E"] \arrow[d, "j^*"] & \Omega^4 S^{10} \arrow[d, "\Omega^4\eta_9"] \\
\Omega^2 S^9\{2\} \arrow[r, "\Omega(\alpha)"] & \Omega^4 S^9
\end{tikzcd}$$

homotopy commutes. Thus by naturality and this last diagram, the composite $\Omega^2(i\partial)\cdot\Omega(\alpha)\cdot j^*\cdot\eta^*$ is homotopic to $\Omega\lambda$ and the proposition follows.

Next, notice that by definition of h in Lemma 3.1, the composite $i^*\cdot h$ is homotopic to ΩH_2 where H_2 denotes the second Hilton–Hopf invariant. Thus by Corollary 7.4 and Proposition 7.5, one has

THEOREM 7.6. The diagram

$$\begin{tikzcd}
\Omega^2 S^5 \arrow[r, "\Omega H_2"] \arrow[dr, "2\phi"'] & \Omega^2 S^9 \arrow[d, "\Omega^2(\overline{v'\eta_6\eta_7})"] \\
 & \Omega S^3
\end{tikzcd}$$

homotopy commutes.

§8. PROOF OF 6.1 AND 1.7

Since ϕ factors through $(\Omega^2 S^9)\{2\}$ which is 5-connected, it follows that ϕ induces the trivial map on π_i, $i \leq 5$. To finish the proof of Proposition 6.1, it suffices to check that $2\phi = 2\Omega^2\bar{\eta}$ in $[\Omega^2 S^5, \Omega S^3]$. The main lemma here is

LEMMA 8.1. *$2\Omega^2\bar{\eta}$ is homotopic to the composite*

$$\Omega^2 S^5 \xrightarrow{\Omega H_2} \Omega^2 S^9 \xrightarrow{\Omega^2[\iota_5, \iota_5]} \Omega^2 S^5 \xrightarrow{\Omega^2\bar{\eta}} \Omega S^3$$

where $[\iota_5, \iota_5]$ is the generator of $\pi_9 S^5$.

By Theorem 7.6, 2ϕ factors as $\Omega^2(\overline{v'\eta_6\eta_7})\cdot\Omega H_2$. Thus Proposition 6.1 follows from Lemma 8.2 which of course relies heavily on Toda's calculations [19].

LEMMA 8.2. *The composite*

$$S^9 \xrightarrow{[\iota_5, \iota_5]} S^5 \xrightarrow{\bar{\eta}} BS^3$$

is homotopic to $\overline{v'\eta_6\eta_7}$.

Proof of 8.2. Consider the composite

$$S^9 \xrightarrow{[\iota_5, \iota_5]} S^5 \xrightarrow{\eta_4} S^4 \xrightarrow{i} BS^3.$$

By [19; p. 43], $[\iota_5, \iota_5] = v_5\eta_8$ and so $\eta_4\cdot[\iota_5, \iota_5] = \eta_4\cdot v_5\eta_8$. By [19; p. 44], $\eta_5 v_4 = v'\eta_6$ and thus we have $\eta_4 v_5 = (Ev')\eta_7$ and $\eta_4 v_5 \eta_8 = (Ev')\eta_7\eta_8$. Clearly $i_*((Ev')\eta_7\eta_8) = v'\eta_6\eta_7$ by commutativity of the diagram

$$\begin{tikzcd}
S^8 \arrow[r, "v'\eta_6\eta_7"] \arrow[d] & S^3 \arrow[d] \arrow[dr, "1"] & \\
\Omega S^9 \arrow[r, "\Omega((Ev')\eta_7\eta_8)"] & \Omega S^4 \arrow[r] & S^3
\end{tikzcd}$$

and so Lemma 8.2 follows.

Proof of 8.1. Notice that composite $S^5 \xrightarrow{[2]} S^5 \xrightarrow{\bar{\eta}} BS^3$ is null-homotopic. Looping twice, we have $\Omega^2(\bar{\eta})\cdot\Omega^2([2]) = 0$ in $[\Omega^2 S^5, \Omega S^3]$. But $\Omega^2([2]) = 2 + \Omega^2 w\cdot\Omega H_2$ by Section 5 of [3] for $w = [\iota_5, \iota_5]$ and $2\Omega^2 w\cdot\Omega H_2 = 0$. Thus $2\Omega^2\bar{\eta} = \Omega^2\bar{\eta}\cdot\Omega^2 w\cdot\Omega H_2$.

Proof of 1.7: Let π: $\Omega S^9 \to BS^3$ denote the composite

$$\Omega S^9 \xrightarrow{\Omega\eta_8} \Omega S^8 \xrightarrow{\Omega E} \Omega^2 S^9 \xrightarrow{\partial} S^4 \xrightarrow{i} BS^3.$$

Consider $(\Omega^2\pi)\cdot\eta^*$. By construction (7.2), λ: $\Omega S^9 \to S^3$ is the composite $\Omega(i\partial)\cdot\eta^*\cdot\Omega^2\eta_9\cdot\Omega E$. Thus $\Omega\lambda$ is homotopic to $(\Omega^2\pi)\cdot\eta^*$ by definition. By Corollary 7.4, $\Omega\lambda$ is homotopic to $\Omega^2(\overline{v'\eta_6\eta_7})$ and by Lemma 8.2, $\overline{v'\eta_6\eta_7}$ is homotopic to $\bar{\eta}\cdot[\iota_5, \iota_5]$.

That η^*: $\Omega^2 S^9 \to \Omega^3 S^9$ is not an H-map is checked at the end of Section 9, and so Proposition 1.7 follows.

§9. PROOF OF 7.3 AND 1.7

We prove Theorem 7.3 which states that λ is homotopic to a loop map. By Lemma 5.1, it suffices to show that λ is an H-map.

The H-deviation of η^* is W^*: $\Omega^2 S^{10} \wedge \Omega^2 S^{10} \to \Omega^3 S^{10}$ by Corollary 5.5. By naturality of the H-deviation, the H-deviation of λ is the composite

$$\Omega S^9 \wedge \Omega S^9 \to \Omega^2 S^{10} \wedge \Omega^2 S^{10} \xrightarrow{W^*} \Omega^3 S^{10} \xrightarrow{\Omega^3\eta_9} \Omega^3 S^9 \xrightarrow{\Omega\partial} \Omega S^4 \xrightarrow{\Omega i} S^3.$$

Again by naturality, the diagram

$$\begin{tikzcd}
\Omega S^9 \wedge \Omega S^9 \arrow[r] \arrow[d, "(\Omega\eta_8)\wedge(\Omega\eta_8)"] & \Omega^2 S^{10} \wedge \Omega^2 S^{10} \arrow[r, "W^*"] \arrow[d, "(\Omega^2\eta_9)\wedge(\Omega^2\eta_9)"] & \Omega^3 S^{10} \arrow[r, "\Omega^3\eta_9"] \arrow[d, "\Omega^3\eta_9"] & \Omega^3 S^9 \arrow[d, "1"] \\
\Omega S^8 \wedge \Omega S^8 \arrow[r] & \Omega^2 S^9 \wedge \Omega^2 S^9 \arrow[r, "W^*"'] & \Omega^3 S^9 \arrow[r, "1"'] & \Omega^3 S^9
\end{tikzcd}$$

homotopy commutes. Thus we have shown

LEMMA 9.1. *The H-deviation of λ is the composite*

$$\Omega S^9 \wedge \Omega S^9 \xrightarrow{\Omega(\eta_8)\wedge\Omega(\eta_8)} \Omega S^8 \wedge \Omega S^8 \to \Omega^2 S^9 \wedge \Omega^2 S^9 \xrightarrow{W^*} \Omega^3 S^9 \xrightarrow{\Omega\partial} \Omega S^4 \xrightarrow{\Omega i} S^3.$$

Consider the map $\Sigma(\Omega\eta_8)\wedge(\Omega\eta_8)$: $\Sigma\Omega S^9 \wedge \Omega S^9 \to \Sigma(\Omega S^8)\wedge(\Omega S^8)$. Since η_8 desuspends and the decomposition $\Sigma\Omega\Sigma X \cong \Sigma \vee X^{[n]}$ is natural for maps of suspensions there is a homotopy commutative diagram

$$\begin{tikzcd}
\Sigma\Omega S^9 \wedge \Sigma\Omega S^9 \arrow[r, "\Sigma(\Omega\eta_9)\wedge(\Omega\eta_9)"] & \Sigma\Omega S^8 \wedge \Omega S^8 \\
\Sigma\left(\bigvee_{i,j\geq 1} S^{8(i+j)}\right) \arrow[r, "\Sigma(\vee\,\eta^{[i+j]})"'] \arrow[u, "\simeq"] & \Sigma\left(\bigvee_{i,j\geq 1} S^{7(i+j)}\right). \arrow[u, "\simeq"]
\end{tikzcd}$$

Since $\eta^{[i+j]}$ is null if $i+j\geq 4$, the adjoint of the H-deviation of λ factors as

$$\begin{tikzcd}[column sep=small]
\Sigma\Omega S^9 \wedge \Omega S^9 \arrow[r] \arrow[d, "\text{pinch}"] & \Sigma\Omega S^8 \wedge \Omega S^8 \arrow[r] & \Sigma\Omega^2 S^9 \wedge \Omega^2 S^9 \arrow[r] & \Sigma\Omega^3 S^9 \arrow[r] & \Sigma\Omega S^4 \arrow[r] & \Sigma S^3 \arrow[d, "\text{evaluate}"] \\
S^{17} \vee S^{24} \vee S^{24} \arrow[r, "\eta_{15}^2 \vee \eta_{21}^3 \vee \eta_{21}^3"'] & S^{15} \vee S^{21} \vee S^{21} \arrow[u] \arrow[rrrr] & & & & BS^3.
\end{tikzcd}$$

Hence, evaluation of the H-deviation rests on computing an element in $[S^{17} \vee S^{24} \vee S^{24}, BS^3] \cong \pi_{17}BS^2 \oplus \pi_{24}BS^3 \oplus \pi_{24}BS^3$.

Once we show that this element is trivial, it follows that the H-deviation vanishes. But notice that an element in $\pi_{24}BS^3$ obtained by *pre*composition with $\eta_n^3 = 4\nu_n$, $n\geq 5$, is divisible by 4. Since $\pi_{24}BS^3 \cong \pi_{23}S^3 \cong \mathbb{Z}/2\oplus\mathbb{Z}/2$ by the main theorem of [15], this element must be zero. (Notice that a non-computational approach here is to use James' result that S^3 has exponent 4.) To finish, it must be checked that the element in $\pi_{17}BS^3 \cong \pi_{16}S^3 \cong \mathbb{Z}/2$ is zero. We need to know the specific form of this element.

The adjoint of this element in $\pi_{16}S^3$ is computed from construction (6.5) and the homotopy commutative diagram

$$\begin{tikzcd}
S^{16} \arrow[r, "\eta_{14}^2"] \arrow[d] & S^{14} \arrow[r, "1"] \arrow[d] & S^{14} \arrow[r, "\overline{[\iota_9,\iota_9]}"] \arrow[d] & \Omega^3 S^9 \arrow[d] & & \\
\Omega S^9 \wedge \Omega S^9 \arrow[r] & \Omega S^8 \wedge \Omega S^8 \arrow[r] & \Omega^2 S^9 \wedge \Omega^2 S^9 \arrow[r] & \Omega^3 S^9 \arrow[r, "\Omega\partial"] & \Omega S^4 \arrow[r] & S^3
\end{tikzcd}$$

where $\overline{[\iota_9,\iota_9]}$ is the adjoint of $[\iota_9,\iota_9]$.

Now by [19; p. 61] $\pi_{17}S^9 \cong \mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2 \cong \{\sigma_9\eta_{16}\}\oplus\{\bar{\nu}_9\}\oplus\{\varepsilon_9\}$, while the kernel of the suspension $\pi_{17}S^9 \to \pi_{18}S^{10}$ is a subgroup of order 2 and with $E(\sigma_9\eta_{16}+\bar{\nu}_9+\varepsilon_9)=0$

[19; p. 65]. Thus $[\iota_9, \iota_9] = \sigma_9\eta_{16} + \bar{\nu}_9 + \varepsilon_9$ and so $[\iota_9, \iota_9]\eta_{17}^2 = (\sigma_9\eta_{16} + \bar{\nu}_9 + \varepsilon_9)\eta_{17}\eta_{18}$. Since all of these elements are suspensions, it follows that

$$[\iota_9, \iota_9]\eta_{17}^2 = (\sigma_9\eta_{16}\eta_{17}\eta_{18}) + (\bar{\nu}_9 + \varepsilon_9)\eta_{17}\eta_{18}.$$

Furthermore $\eta_9 \cdot \sigma_{10} = \bar{\nu}_9 + \varepsilon_9$ and $\eta_n \cdot \sigma_{n+1} = \sigma_n \cdot \eta_{n+7}$ if $n \geq 10$ by Lemma 6.4 [19; p. 54]. Combining this information we have the equation

$$[\iota_9, \iota_9]\eta_{17}^2 = (\sigma_9\eta_{16}\eta_{17}\eta_{18}) + \eta_9\eta_{10}\eta_{11}\sigma_{12} = [\sigma_9(4\nu_{16})] + [(4\nu_9)\sigma_{12}].$$

Since σ_{12} desuspends,

$$[\iota_9, \iota_9]\eta_{17}^2 = 4(\sigma_9\nu_{16} + \nu_9\sigma_{12})$$

which projects to zero in $\pi_{17}BS^3 \cong \mathbb{Z}/2$. Thus 4.2 follows.

It is worth pointing out that $\pi_{19}S^9 \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$ where $(4\nu_9)\sigma_{12} = 0$ and $\sigma_9\nu_{16}$ is of order 8 [19; pp. 66 and 72]. Thus the composite $\Omega S^9 \to \Omega^3 S^9$ in our factorization of λ is not an H-map. Hence $\eta^*\colon \Omega^2 S^9 \to \Omega^3 S^9$ is not an H-map as was claimed in Proposition 1.7.

Finally, it is worth remarking that $[\iota_{4n+1}, \iota_{4n+1}] \cdot \eta_{8n+1}^2$ is divisible by 4.

§10. $2\Omega^2\bar{\eta}$, EXAMPLE 3.3

By Lemma 8.1, $2\Omega^2\bar{\eta}$ is homotopic to $(\Omega^2\bar{\eta}) \cdot (\Omega^2[\iota_5, \iota_5]) \cdot (\Omega H_2)$. We shall show

PROPOSITION 10.1. *$2\Omega^2\bar{\eta}$ restricted to the 7-skeleton of $\Omega^2 S^5$ is not null.*

Remark 10.2. $4\Omega^2\bar{\eta} = 0$ in $[\Omega^2 S^5, \Omega S^3]$ by Example 1.3 of [4]. That $2\Omega^2\bar{\eta} \neq 0$ is essentially Example 1.3 of [4].

In order to prove Proposition 10.1 we must record the 7-skeleton of $\Omega^2 S^5$. Let Y denote the homotopy cofibre of $\alpha\colon S^6 \to S^3 \vee S^6$ where α is given by ν' on S^3 and is degree 2 on S^6.

LEMMA 10.3. *There is a map of Y to $\Omega^2 S^5$ which induces a homology isomorphism in dimensions ≤ 8.*

Proof: Let $E^2\colon S^3 \to \Omega^2 S^5$ be the double suspension. Consider $\Omega E\colon \Omega S^4 \to \Omega^2 S^5$. Since ΩS^4 is homotopy equivalent to $S^3 \times \Omega S^7$ the 6-dimensional homology class in $\Omega^2 S^5$ is spherical. Let $\xi\colon S^6 \vee S^3 \to \Omega^2 S^5$ denote the map obtained from E^2 and this splitting of ΩS^4.

Let X denote the homotopy theoretic fibre of ξ. A calculation with the homology Serre spectral sequence shows that there is a map $i\colon S^6 \to X$ which is a homology isomorphism in dimensions ≤ 7.

By the Serre exact sequence the composite $S^6 \xrightarrow{j} S^6 \vee S^3 \xrightarrow{\xi} \Omega^2 S^5$ induces a long exact sequence in homotopy and homology (with $\mathbb{Z}_2$-coefficients) through dimension 7. Since $H_6(\Omega^2 S^5, \mathbb{Z}) \cong \mathbb{Z}/2$, the map $S^6 \xrightarrow{j} S^6 \vee S^3$ is degree 2 on S^6.

On homotopy, there is an exact sequence

$$\pi_6 S^6 \to \pi_6(S^6 \vee S^3) \to \pi_6 \Omega^2 S^5 \to 0$$

which is given by

$$\mathbb{Z} \xrightarrow{j_*} \mathbb{Z} \oplus \mathbb{Z}/4 \xrightarrow{\xi_*} \mathbb{Z}/8 \to 0.$$

Let c denote a generator for the left-hand $\mathbb{Z}$, b a generator for $\mathbb{Z}$ in $\mathbb{Z} \oplus \mathbb{Z}/4$, ν' a generator for $\mathbb{Z}/4$, and ν a generator for $\mathbb{Z}/8$.

Clearly ν' may be chosen such that $\xi_*(\nu') = 2\nu$ since ξ_* restricted to $\mathbb{Z}/4$ is given by the composite $E_*^2\colon \pi_6 S^3 \to \pi_6 \Omega^2 S^5$ [19; p. 42]. Furthermore $\xi_*(b)$ must generate $\mathbb{Z}/8$ since $\xi_*(b) \equiv 1(2)$ because ξ_* is an epimorphism. Assume that $\xi_*(b) = -\nu$. [b may be chosen to have this property since we are free to multiply b by any odd integer and work over $\mathbb{Z}_{(2)}$.] Finally, we may assume that c is chosen such that $j(c) = 2b + x\nu'$.

Since $\xi_* j_* = 0$, it follows that

$$0 = \xi_* \cdot j_*(c) = \xi_*(2b + xv') = -2v + 2xv.$$

Thus $2x \equiv 2(8)$ and so $x \equiv 1(4)$. (Recall that $x \in \mathbb{Z}/4$.) Thus $j_*(c) = 2b + v'$.

Since $S^6 \xrightarrow{j} S^6 \vee S^3 \xrightarrow{\xi} \Omega^2 S^5$ is a cofibration through dimension 8, the lemma follows.

Proof of 10.1: Consider the composite

$$Y \to \Omega^2 S^5 \xrightarrow{\Omega h_2} \Omega^2 S^9 \xrightarrow{\Omega^2[\iota_5, \iota_5]} \Omega^2 S^5 \xrightarrow{\Omega^2 \bar{\eta}} \Omega S^3$$

and notice that it is given by

$$Y \xrightarrow{\text{pinch}} S^7 \to \Omega^2 S^5 \xrightarrow{\Omega^2 \bar{\eta}} \Omega S^3.$$

Adjoint and consider

$$\Sigma^2 Y \xrightarrow{\text{pinch}} S^9 \xrightarrow{[\iota_5, \iota_5]} S^5 \xrightarrow{\bar{\eta}} BS^3.$$

We show that this composite is non-zero.

Since

$$S^6 \to S^6 \vee S^3 \to Y \xrightarrow{\text{pinch}} S^7$$

is a cofibre sequence, the Barratt–Puppe sequence insures a long exact sequence of groups

$$\cdots \leftarrow [S^8 \vee S^5, Z] \leftarrow [\Sigma^2 Y, Z] \leftarrow [S^9, Z] \leftarrow [S^9 \vee S^6, Z] \leftarrow \cdots$$

for any pointed space Z.

Now observe that $\bar{\eta} \cdot [\iota_5, \iota_5]$ is adjoint to $v'\eta_6^2$ by Lemma 8.2. Thus the image of the non-trivial element $v'\eta_6^2$ of $[S^9, BS^3]$ in $[\Sigma^2 Y, BS^3]$ is zero if and only if it is in the image of $(E^3 j)^*$: $[S^9 \vee S^6, BS^3] \to [S^9, BS^3]$. We claim that $(E^3 j)^*$ is the zero map and so Example 3.3 follows.

Notice that $[S^9 \vee S^6, BS^3] \cong \pi_8 S^3 \oplus \pi_5 S^3$ as a group and the induced map $(E^3 j)^*$ is a group homomorphism. $(E^3 j)^*$ on the generator of $\pi_9 BS^3$ is given by the composite

$$S^9 \xrightarrow{E^3 j} S^9 \vee S^6 \xrightarrow{\text{project}} S^9 \to BS^3.$$

Since $E^3(j)$ is degree 2 on the 9-sphere, this last map is zero. Next, observe that $(E^3 j)^*$ on the generator of $\pi_6 BS^3$ is given by the composite

$$S^9 \xrightarrow{E^3 j} S^9 \vee S^6 \xrightarrow{\text{project}} S^6 \to BS^3.$$

Thus, it suffices to consider

$$S^8 \xrightarrow{E^2 v'} S^5 \xrightarrow{\eta_4} S^4 \xrightarrow{\eta_3} S^3$$

which is zero [19; p. 44]. Thus the example follows.

§11. PROOFS OF 6.2 AND 6.4

We prove Proposition 6.2 which states that the maps ϕ and g of Proposition 6.1 and Theorem 3.2 respectively cannot be H-maps. In particular, we show

PROPOSITION 11.1. *Let* $f: \Omega^2 S^5 \to \Omega S^3$ *be an H-map which is onto* π_3. *Then* $2f \neq 0$ *in the group* $[\Omega^2 S^5, \Omega S^3]$.

Proof: Observe that since ΩS^3 is homotopy abelian, the sum of two H-maps is again an H-map. Thus $\rho = (\Omega^2 \bar{\eta}) - f$ is an H-map which is trivial on π_3.

Since the composite $\Omega S^4 \xrightarrow{\Omega E} \Omega^2 S^5 \xrightarrow{\rho} \Omega S^3$ is null-homotopic, ρ is trivial on the 6-skeleton of $\Omega^2 S^5$.

Restrict ρ to Y, the 7-skeleton of $\Omega^2 S^5$. By the above paragraph, $\tau = \rho|_Y$ factors through the 7-sphere: The diagram

$$\begin{tikzcd} Y \arrow[r, "\text{pinch}"] \arrow[dr, "\tau"'] & S^7 \arrow[d] \\ & \Omega S^3 \end{tikzcd}$$

homotopy commutes. Since $\pi_8 S^3 \cong \mathbb{Z}/2$, it follows that $2(\tau)$ is null.

If $2f = 0$ in $[\Omega^2 S^5, \Omega S^3]$, then $2\rho = 2(\Omega^2 \bar{\eta})$ which by Proposition 10.1 is non-trivial on Y. The above paragraph gives that 2ρ restricted to Y is null. This is a contradiction.

Thus g of Theorem 2.2 cannot be an H-map. If ϕ of Proposition 6.1 were an H-map, then g would be an H-map since $g = \Omega^2(\bar{\eta}) - \phi$; Proposition 6.2 follows.

We next check Lemma 6.4 which states that $(\Omega^2[2]) \cdot \Omega E = 2\Omega E$ in $[\Omega S^n, \Omega^2 S^{n+1}]$. If n is even, then $\Omega^2[2] = 2 + \Omega^2 w \cdot \Omega H_2$ in $[\Omega^2 S^{n+1}, \Omega^2 S^{n+1}]$ by Section 5 of [3] where H_2 is the second Hilton–Hopf invariant. By the remarks there, a similar but more complicated formula involving a third Hopf invariant is satisfied when n is odd. Since $\Omega H_i \cdot \Omega E = 0$ if $i \geq 2$, it follows that $\Omega^2[2] \cdot \Omega E = 2\Omega E$ in $[\Omega S^n, \Omega^2 S^{n+1}]$ and so Lemma 6.4 follows.

§12. PROOF OF 4.2

We prove Lemma 4.2 which states that

(i) $\gamma: \Omega^2 S^3 \langle 3 \rangle \times W_2 \to (\Omega^2 S^5)\{2\}$ induces an isomorphism on the module of primitives for mod-2 homology in degrees 2 and 5, and

(ii) the mod-2 homology of $\Omega^2 S^3 \langle 3 \rangle \times W_2$ is isomorphic to that of $(\Omega^2 S^5)\{2\}$ as a coalgebra over the Steenrod algebra.

We first prove (i). Consider $\bar{\rho}: \Omega^2 S^9 \to \Omega S^5$ of Theorem 4.1 and thus obtain a choice of map ρ: $(\Omega^3 S^9)\{2\} \to (\Omega^2 S^5)\{2\}$ giving a morphism of fibrations

$$\begin{tikzcd} \Omega^4 S^9 \arrow[r, "\Omega^2 \bar{\rho}"] \arrow[d, "2"'] & \Omega^3 S^5 \arrow[d, "2"] \\ \Omega^4 S^9 \arrow[r, "\Omega^2 \bar{\rho}"] \arrow[d, "j^*"'] & \Omega^3 S^5 \arrow[d, "j^*"] \\ (\Omega^3 S^9)\{2\} \arrow[r, "\rho"] & (\Omega^2 S^5)\{2\} \end{tikzcd}$$

Next, recall that the class in $H_6(\Omega^2 S^5)$ is spherical. In particular, the 8-skeleton of $\Omega^2 S^5$ is given by the space Y in Lemma 10.3.

LEMMA 12.1. *Let $v: S^6 \to \Omega^2 S^5$ be any map which induces a non-zero map on H_6. Then v generates $\pi_6 \Omega^2 S^5 \cong \mathbb{Z}/8$.*

Proof: Since v induces an epimorphism on $H_6(\Omega^2 S^5; \mathbb{Z}) \cong \mathbb{Z}/2$, v is not divisible by 2.

Thus by Lemmas 4.2 and 12.1, the map $\Omega\bar{\rho}$: $\Omega^3 S^9 \to \Omega^2 S^5$ induces an isomorphism on $H_6(;\mathbb{Z}/2)$. Hence ρ induces a monomorphism on $H_5(;\mathbb{Z}/2)$.

Since $H_*((\Omega^2 S^5)\{2\}; \mathbb{Z}/2) \cong H_*(\Omega^2 S^5; \mathbb{Z}/2) \otimes H_*(\Omega^3 S^5; \mathbb{Z}/2)$ as a Hopf algebra, there is exactly one primitive in degrees 2, 3, 4 and 5. By the preceding paragraph, γ induces an isomorphism on the module of primitives in degree 5. By definition of γ and Lemma 3.1, γ induces an isomorphism in degree 2. Thus Lemma 4.2(i) follows.

To prove Lemma 4.2(ii), we recall the action of the Steenrod algebra on the mod-2 homology of $(\Omega^2 S^5)\{2\}$ and $\Omega^2 S^3\langle 3\rangle \times W_2$. For the remainder of this section, all homology groups are taken with $\mathbb{Z}/2$-coefficients.

By Section 2 of [2], $H_*(\Omega^2 S^5)\{2\} \cong \mathbb{Z}/2[x_2, x_3, \bar{Q}_1^a x_3, Q_1^b Q_2^c x_2]$ as a Hopf algebra for $a \geq 1$ and $b + c \geq 1$ with the degree of x_i given by i. The action of the Steenrod algebra is given by

(1) $Sq_*^1 x_3 = x_2$,

(2) the Nishida relations for $Q_1^b Q_2^c x_2$, and

(3) $Sq_*^{2^i} \bar{Q}_1^a x_3 = \begin{cases} (\bar{Q}_1^{a-1} x_3)^2 & \text{if } i = 0 \\ 0 & \text{if } i > 0, \end{cases}$

with $Q_1^0 x_3 = x_3$.

To describe the coalgebra structure of $H_*(\Omega^2 S^3\langle 3\rangle \times W_2)$, first recall that $H_*\Omega^2 S^3\langle 3\rangle \cong \mathbb{Z}/2[x_2, Q_1^a x_1]$ as a Hopf algebra for $a \geq 1$. Furthermore, the action of the Steenrod operations on $H_*\Omega^2 S^3\langle 3\rangle$ is given by [5; p. 214]

$$Sq_*^{2^i} Q_1^a x_1 = \begin{cases} (Q_1^{a-1} x_1)^2 & \text{if } i = 0, a > 1, \\ x_2 & \text{if } i = 0, a = 1, \\ 0 & \text{if } i > 0. \end{cases}$$

The action of the Steenrod operations on W_2 follows from the fact that $\Omega^2 S^5 \to W_2$ induces an epimorphism in homology. Thus as a *coalgebra*, over the Steenrod algebra, $H_* W_2$ is isomorphic to the Hopf algebra $\mathbb{Z}/2[Q_1^b Q_2^c x_2]$, $b + c \geq 1$, where the Steenrod operations are given by the Nishida relations (with $x_2^i = 0$ for $i \geq 1$). Thus by inspection, $H_*(\Omega^2 S^5)\{2\}$ and $H_*(\Omega^2 S^3\langle 3\rangle \times W_2)$ are isomorphic as coalgebras over the Steenrod algebra.

§13. PRIMITIVES IN $H_*\Omega_0^3 S^3$ AND THE PROOF OF 1.8

Throughout Sections 13–15, all homology groups are taken with $\mathbb{Z}/2$-coefficients. Recall that $H_*\Omega_0^3 S^3 \cong \mathbb{Z}/2[Q_1^a Q_2^b[1] * [-2^{a+b}]]$ where $Q_1^a Q_2^b[1] = Q_1 \underset{a}{\cdots} Q_1 Q_2 \underset{b}{\cdots} Q_2[1]$ [5; p. 226]. We shall omit the notation "$*[-2^{a+b}]$" and the symbol "$Q_1^a Q_2^b[1]$" will denote $(Q_1^a Q_2^b[1]) * [-2^{a+b}]$.

PROPOSITION 13.1. *There is at most one primitive in any fixed degree of $H_*\Omega_0^3 S^3$.* A basis for primitives, $PH_*\Omega_0^3 S^3$, is given by the 2^i-th powers of the elements

(i) $Q_1^a[1]$, $a > 0$, and

(ii) $Q_1^a Q_2^b[1] + (Q_1^{a+b}[1])(Q_1^b[1])^{2^a}$, $a, b > 0$.

Clearly $PH_*\Omega_0^3 S^3$ is an $\mathscr{A}^{0p}$ module in the usual way ($Sq^i(x) = Sq_*^i x$) where $\mathscr{A}$ is the mod-2 Steenrod algebra. Let $\zeta: PH_*\Omega_0^3 S^3 \to PH_*\Omega_0^3 S^3$ be any map which commutes with the $\mathscr{A}^{0p}$-action.

PROPOSITION 13.2. *If ζ is an isomorphism in degrees less than or equal to 2, then ζ is an isomorphism.*

This last proposition will be proven in Section 15. Let $f: \Omega_0^3 S^3 \to \Omega_0^3 S^3$ be *any* map of spaces.

PROPOSITION 1.8. *If f_* is an isomorphism on π_i for $i \leq 2$, then f is a homotopy equivalence.*

Proof of 1.8. If f_* induces an isomorphism on π_i for $i \leq 2$, then f_* induces an isomorphism on $PH_*\Omega_0^3 S^3$ in degrees ≤ 2. Thus f_* restricted to $PH_*\Omega_0^3 S^3$ is an isomorphism by Proposition 13.2 and so f_* is an isomorphism. Since $\Omega_0^3 S^3$ is a connected H-space, Proposition 1.8 follows from the generalized Whitehead theorem for maps between connected H-spaces.

We shall finish this section by supplying a proof of Proposition 13.1 by first recording the degrees of the generators $Q_1^a Q_2^b[1]$.

LEMMA 13.3. *The degree of* $Q_1^a Q_2^b[1]$ *is given by*

$$|Q_1^a Q_2^b[1]| = 2^a - 1 + 2^a(2^{b+1} - 2).$$

Proof. The degree of $Q_2^b[1]$ is $2^{b+1} - 2$ by induction and the degree of $Q_1^a x$ is $2^a - 1 + 2^a(|x|)$ by induction.

To compute $PH_*\Omega_0^3 S^3$ we need the exact sequence of Milnor–Moore in Proposition 2.13 of [14] given by

$$0 \to P(k\xi(A)) \to P(A) \to Q(A) \to Q(k\lambda(A)) \to 0$$

where A is a connected Hopf algebra with both commutative multiplication and commutative comultiplication over a field k of characteristic $p \neq 0$, ξ is the p^{th} power map, kW is the vector space generated by W and $k\lambda A$ is $(k\xi(A^*))^*$. First of all, they show that $P(A) \to Q(A)$ is an isomorphism in degrees $\equiv 1(2)$. Apply this to $A = H_*\Omega_0^3 S^3$.

Thus by the above exact sequence and Lemma 13.3, the odd degree primitives are in degrees $2^a - 1 + 2^a(2^{b+1} - 2)$ for $a \geq 1$ and $b \geq 0$. Again by Lemma 13.3, the module of indecomposables concentrated in even degrees is zero in degrees $\neq 2^{b+1} - 2$ and is $\mathbb{Z}/2$ in degrees $2^{b+1} - 2$ for $b \geq 1$. We shall show

LEMMA 13.4. *The map* $Q(A) \to Q(k\lambda(A))$ *is an isomorphism in degrees* $2^{b+1} - 2$.

COROLLARY 13.5. *In odd degrees* $P(A) \to Q(A)$ *is an isomorphism. In even degrees* $P(k\xi(A) \to P(A)$ *is an isomorphism. Thus* $P(A)$ *is spanned by the* 2^i*-th power of odd degree primitives.*

To do these calculations the diagonal map Δ must be written down.

LEMMA 13.6.

(*i*) $Q_1^a[1]$ *is primitive.*

(*ii*) $\Delta Q_2^b[1] = Q_1^b[1] \otimes Q_1^b[1] + Q_2^b[1] \otimes 1 + 1 \otimes Q_2^b[1] + \Sigma y \otimes y'$ *for* y *or* y' *decomposable.*

(*iii*) $\Delta Q_1^a Q_2^b[1] = Q_1^a Q_2^b[1] \otimes 1 + Q_1^{a+b}[1] \otimes (Q_1^b[1])^{2^a} + (Q_1^b[1]^{2^a} \otimes Q_1^{a+b}[1] + 1 \otimes Q_1^a Q_2^b[1]$.

The proof is given in Section 14.

Using Lemma 13.6, we prove Lemma 13.4 and then Proposition 13.1.

Proof of 13.4. Recall that $k\lambda(A) = (k\xi(A^*))^*$. $Q(k\lambda A)$ is non-zero provided $P(k\xi(A^*))$ is non-zero since the module of indecomposables is dual to the module of primitives. We shall study this in A^*.

A basis for $A = H_*\Omega_0^3 S^3$ is given by the basis of monomials. Consider the dual basis to the monomial basis. Then the dual basis element $(Q_1^a[1])^*$ is primitive in the dual basis since $Q_1^a[1]$ is indecomposable in $H_*\Omega_0^3 S^3$.

Since $\Delta Q_2^a[1] = Q_1^a[1] \otimes Q_1^a[1] + \text{others}$, it follows that $((Q_1^a[1])^*)^2 = (Q_2^a[1])^* + \text{others}$ in A^*. In particular $\xi((Q_1^a[1])^*) \neq 0$. But $\xi((Q_1^a[1])^*)$ is primitive since $Q_1^a[1]$ is indecomposable. Thus $P_{2^{a+1}-2}(k\xi(A^*)) \neq 0$ and so $Q_{2^{a+1}-2}(k\lambda A) \neq 0$.

Since $Q_{2^{a+1}-2}(A) = \mathbb{Z}/2$ by Lemma 13.3 and $Q_{2^{a+1}-2}(A) \to Q_{2^{a+1}-2}(k\lambda A)$ is onto a non-zero module, it follows that these two modules are isomorphic.

We next give

Proof of 13.1. By Corollary 13.5 and Lemma 13.3 the primitives in $H_*\Omega_0^3 S^3$ are given by the 2^i-th powers of elements in degrees $2^a - 1 + 2^a(2^{b+1} - 2)$ for $a > 0$. We claim that there is at most one of these in any degree.

It suffices to check that if $2^N[2^a - 1 + 2^a(2^{b+1} - 2)] = 2^M[2^c - 1 + 2^c(2^{d+1} - 2)]$ for a,

$c > 0$, then $N = M$, $a = c$ and $b = d$. First notice that $N = M$ because $2^a - 1 + 2^a(2^{b+1} - 2)$ and $2^c - 1 + 2^c(2^{d+1} - 2)$ are odd. Thus it suffices to check when

$$2^a - 1 + 2^a(2^{b+1} - 2) = 2^c - 1 + 2^c(2^{d+1} - 2)$$

or equivalently

$$2^a(2^{b+1} - 1) = 2^c(2^{d+1} - 1).$$

It follows that $a = c$ and $b = d$.

To prove the second part of Proposition 13.1 it thus suffices to exhibit one primitive in degree $2^a - 1 + 2^a(2^{b+1} - 2)$. By Lemma 13.6(i) and (iii), it follows that the listed elements in Proposition 13.1 are primitive: $Q_1^a[1]$ is primitive by Lemma 13.6(i). Furthermore if $\bar{\Delta}$ denotes the reduced diagonal, it follows that

$$(1)\ \bar{\Delta}(Q_1^{a+b}[1])(Q_1^b[1])^{2^a} = Q_1^{a+b}[1] \otimes (Q_1^b[1])^{2^a} + (Q_1^b[1])^{2^a} \otimes Q_1^{a+b}[1]$$

and

$$(2)\ \bar{\Delta}Q_1^a Q_2^b[1] = Q_1^{a+b}[1] \otimes (Q_1^b[1])^{2^a} + (Q_1^b[1])^{2^a} \otimes Q_1^{a+b}[1]$$

by Lemma 13.6(ii). Thus $Q_1^a Q_2^b[1] + (Q_1^{a+b}[1])(Q_1^b[1])^{2^a}$ is primitive.

Wellington gives an inductive method to compute primitives in $H_*\Omega_0^n S^n$ [21]. Although the proof of Proposition 13.1 is mildly redundant, we include the details for the convenience of the reader.

§14. THE DIAGONAL MAP AND THE PROOF OF 13.6

Some facts about $\Delta Q_1^a Q_2^b[1]$ are recorded in this section. A proof of Lemma 13.6 is given using the following Adem relations.

LEMMA 14.1. *(i)* $Q_2 Q_1 x = 0$, *(ii)* $Q_1(x^2) = 0$, *and (iii)* $Q_2(x^2) = (Q_1 x)^2$ *for any class* x.

Proof. Inspection

To compute the diagonal for $Q_1^a Q_2^b[1]$, notice that

$$\Delta Q_1^a Q_2^b[1] = \sum Q_{i_1} \ldots Q_{i_a} Q_{j_1} \ldots Q_{j_b}[1] \otimes Q_{i'_1} \ldots Q_{i'_a} Q_{j'_1} \ldots Q_{j'_b}[1]$$

where $i_t + i'_t = 1$ and $j_s + j'_s = 2$ since $\Delta[1] = [1] \otimes [1]$.

We use this information to give the

Proof of 13.6. If $b = 0$, observe that all terms in $\Delta Q_1^a[1]$ except $Q_1^a[1] \otimes Q_0^a[1] + Q_0^a[1] \otimes Q_1^a[1]$ vanish, and so Lemma 13.6(i) follows.

Next notice that

$$\Delta Q_2^b[1] = \sum Q_{j_1} \ldots Q_{j_b}[1] \otimes Q_{j'_1} \ldots Q_{j'_b}[1].$$

All terms with $\ldots Q_2 Q_1 \ldots$ are zero by Lemma 14.1(i). All terms with $\ldots Q_2 Q_0 \ldots$ are decomposable by Lemma 14.1(iii). All terms with $\ldots Q_1 Q_0 \ldots$ are zero by Lemma 14.1(ii). It follows that

$$\Delta Q_2^b[1] = Q_2^b[1] \otimes Q_0^b[1] + Q_1^a[1] \otimes Q_1^a[1] + Q_0^b[1] \otimes Q_2^b[1] + \sum y \otimes y'$$

where y and y' are decomposable.

We prove 10.7(iii) similarly. First, $\Delta Q_1^a Q_2^b[1] = \Sigma Q_{i_1} \ldots Q_{i_a}\ Q_{j_1} \ldots Q_{j_b}[1] \otimes Q_{i'_1} \ldots Q_{i'_a} Q_{j'1} \ldots Q_{j'_b}\ [1]$ with $0 \le i_s \le 1$. Terms of the form $\ldots Q_1 Q_0 \ldots$ vanish by Lemma 14.1 (ii) and so we may assume that

$$\Delta Q_1^a Q_2^b[1] = \sum Q_1^a Q_{j_1} \ldots Q_{j_b}[1] \otimes Q_0^a Q_{j'_1} \ldots Q_{j'_b}[1] + \sum Q_0^a Q_{j_1} \ldots Q_{j_b}[1] \otimes Q_1^a Q_{j_1} \ldots Q_{j'_b}[1].$$

An inspection of the terms in the diagonal together with the Adem relations in Lemma 14.1 gives

$$\Delta Q_1^a Q_2^b[1] = Q_1^a Q_2^b[1] \otimes 1 + Q_1^{a+b}[1] \otimes (Q_1^b[1])^{2^a} + (Q_1^b[1]^{2^a} \otimes Q_1^{a+b}[1] + 1 \otimes Q_1^a Q_2^b[1].$$

§15. PROOF OF 13.2

Assume that $\zeta\colon PH_*\Omega_0^3 S^3 \to PH_*\Omega_0^3 S^3$ is any map which commutes with the $\mathscr{A}^{0p}$-action and which is an isomorphism in degrees ≤ 2. Then Proposition 13.2 states that ζ is an isomorphism.

It is worthwhile remarking that $PH_*\Omega_0^3 S^3$ is a rather nice $\mathscr{A}^{0p}$-module.

Our main calculations involve the Steenrod operations on the element

$$x_b = Q_1 Q_2^b[1] + (Q_1^{b+1}[1]\,(Q_1^b[1])^2, \quad b > 0.$$

In particular, let $Sq^{I_b} = (Sq^2)(Sq^4)(Sq^{2^{b-1}})(Sq^{2^b})$.

LEMMA 15.1. $Sq_*^{I_b}(x_b) = Q_1^{b+1}[1]$ if $b > 0$.

Proof. We check that

(a) $Sq_*^{I_b} Q_1 Q_2^b[1] = Q_1^{b+1}[1]$ and

(b) $Sq_*^{I_b}[(Q_1^{b+1}[1])(Q_1^b[1]^2] = 0$.

To check (a), it suffices to see that $Sq_*^{2k} Q_1 x = Q_1 Sq_*^k x$ which follows from the Nishida relations. Indeed then

$$\begin{aligned} Sq_*^{2^b} Sq_*^{2^{b-1}} \ldots Sq_*^4 Sq_*^2 Q_1 Q_2^b[1] &= Sq_*^{2^b} \ldots Sq_*^4 Q_1 Q_1 Q_2^{b-1}[1] \\ &= Sq_*^{2^b} \ldots Sq_*^8 Q_1 Sq_*^2 Q_1 Q_2^{b-1}[1] \\ &= Sq_*^{2^b} \ldots Sq_*^8 Q_1 Q_1 Q_2^{b-2}[1] \\ &\;\;\vdots \\ &= Q_1^{b+1}[1]. \end{aligned}$$

To check (b), *it suffices to see that* $Sq_*^1 Q_1^b[1] = (Q_1^{b-1}[1])^2$ if $b > 1$ and $Sq_*^{2^i} Q_1^b[1] = 0$ if $i > 1$ which again follows by inspection of the Nishida relations:

$$\begin{aligned} Sq_*^{2^b} \ldots Sq_*^2 (Q_1^{b+1}[1])(Q_1^b[1])^2 &= Sq_*^{2^b} \ldots Sq_*^4 (Q_1^{b+1}[1])(Q_1^{b-1}[1])^4 \\ &= Sq_*^{2^b} \ldots Sq_*^8 (Q_1^{b+1}[1])(Q_1^{b-2}[1])^8 \\ &\;\;\vdots \\ &= Sq_*^{2^b}(Q_1^{b+1}[1])(Q_1[1])^{2^b} \\ &= 0, \end{aligned}$$

and the lemma follows.

To do the next calculation observe that

(i) $Sq_*^1 x_b = Sq_*^1 (Q_1 Q_2^b[1] + (Q_1^{b+1}[1])(Q_1^b[1])^2) = (Q_1^b[1])^4$

since $b > 0$. Set $J_b = (1, 2, \ldots, 2^{b-2})$ and observe that

(ii) $Sq_*^{J_b} Q_1^b[1] = (Q_1[1])^{2^{b-1}}$ if $b > 1$

(where $Sq_*^{J_b} = Sq_*^{2^{b-2}} \ldots Sq_*^2 Sq_*^1$) and

(iii) $Sq_*^{4J_b}(Q_1^b[1])^4 = (Q_1[1])^{2^{b+1}}$.

Proof of 13.2. If v is an element of $PH_*\Omega_0^3 S^3$, then $\zeta(v) = \alpha v$ for some scalar depending on

v since there is at most one primitive in any fixed degree by Proposition 13.1. We check that $\alpha = 1$.

The first stage of the proof is to check Proposition 13.2 for $v = z_N = (Q_1[1])^{2^N}$ by induction on N. The hypotheses of Proposition 13.2 give the result for $N = 0, 1$. Assume that result is correct for $N \leq k$ and check it for $N = k+1$ by induction with $k \geq 1$: Consider $x_k = Q_1 Q_2^k[1] + (Q_1^{k+1}[1])(Q_1^k[1])^2$. Then

(1) $Sq_*^1 x_k = (Q_1^k[1])^4$ by (i), and

(2) $Sq_*^{4J_k} Sq_*^1 x_k = (Q_1[1])^{2^{k+1}}$ by (iii), if $k > 1$.

Furthermore

(3) $Sq_*^{I_k} x_k = Q_1^{k+1}[1]$ by Lemma 15.1,

and so

(4) $Sq_*^{J_{k+1}} Sq_*^{I_k} x_k = (Q_1[1])^{2^k}$.

If $k = 1$, then $\zeta(x_k) = x_k$ by (4). Thus $\zeta(z_2) = z_2$ by (1). Hence it suffices to assume that $k > 1$. But then $\zeta(x_k) = x_k$ by (4) together with the induction bypothesis. Then $\zeta(z_{k+1}) = z_{k+1}$ by (2).

Since $\zeta(z_k) = z_k$ for all k,

(5) $Sq_*^{2^M J_{k+1}} Sq_*^{2^M I_k} (x_k)^{2^M} = (Q_1[1])^{2^{k+M}}$ and

(6) $Sq_*^{2^M J_k} (Q_1^k[1])^{2^M} = (Q_1[1])^{2^{k+M-1}}$ if $k > 1$,

it follows that $\zeta(x) = x$ for x in $PH_*\Omega_0^3 S^3$.

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